Nonlinear Theory for Plates and Shells Including the Effects of Transverse Shearing

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Nonlinear strain displacement relations for three-dimensional elasticity are determined in orthogonal curvilinear coordinates. To develop a two-dimensional theory, the displacements are expressed by trigonometric series representation through the thickness. The nonlinear strain-displacement relations are expanded into a series that contains all first- and second-degree terms. In the series for the displacements only the first few terms are retained. Insertion of the expansions into the three-dimensional virtual work expression leads to nonlinear equations of equilibrium for laminated and thick plates and shells that include the effects of transverse shearing. Equations of equilibrium and buckling equations are derived for flat plates and cylindrical shells. The shell equations reduce to conventional transverse shearing shell equations when the effects of the trigonometric terms are omitted and to classical shell equations when the trigonometric terms are omitted and the shell is assumed to be thin. Numerical results are presented for the buckling of a thick simply supported flat rectangular plate in longitudinal compression.

Introduction

THE objective in developing a two-dimensional shell theory is to predict stresses and deformations that are in good agreement with those predicted by three-dimensional elasticity theory. Even when extensive computer facilities are available, there is a need for a two-dimensional theory because it provides a more tractible approach than three-dimensional elasticity. Shortcomings in existing shell theories are associated with initial imperfections, inextensional behavior, choice of nonlinear terms, and choice of deep shell terms. Also, special problems arise when shell theory is used for two-dimensional representation of three-dimensional phenomena, such as those in a boundary layer near a free edge, near a concentrated load or between laminas in a laminated composite shell.

For thin plates and shells of homogeneous material, classical plate or shell theory predicts in-plane stresses and deformations comparable to those given by three-dimensional elasticity. Conventional transverse shearing theory makes similar predictions for sandwich construction and for thicker plates and shells of homogeneous material. Transverse stresses are generally small compared to the largest in-plane stress; however, they can be important when the structure is relatively weak in the transverse direction and when the structural response is sensitive to the transverse stiffness as in buckling and the higher modes of vibration.

A purpose of the present paper is to develop an accurate nonlinear two-dimensional theory for laminated and thick plates and shells that can predict the in-plane stresses as well as transverse direct stress and transverse shearing stresses. The need for more accurate analysis for laminated plates and shells has led to the appearance of a number of theories (e.g., Refs. 1-10). The theories that have appeared are all linear and use power series for displacements. Some unnecessarily satisfy natural boundary conditions at the plate surface and some apply arbitrary corrections to the transverse shear stiffness. A theory of shells, almost by definition, deals with variables defined only on a reference surface, normally the middle surface. Reference 11 presents a derivation of plate and shell equations that starts with linear strain-displacement relations and the first few terms of a through-the-thickness power series for the displacements by use of a three-dimensional variational theorem. The displacements are defined not only on the reference surface, but throughout the plate or shell. The present derivation of plate and shell equations is similar to that of Ref. 11, except that nonlinear strain-displacement relations are used and trigonometric terms are used for the displacements in addition to the initial terms of a power series through the thickness. The present paper presents the derivation of nonlinear plate and cylindrical shell equations and equations for buckling of plates and cylindrical shells under simple loadings. The approach used here is the same as that in Ref. 12 deriving a theory for the cylindrical bending of plates. Convergence is examined in Ref. 12 for the various stresses and deformations including the transverse stresses and comparisons are made with linear elasticity. For laminated plates with only a few layers, convergence is improved if the potential energy method can be used for deflections and in-plane stresses and the complementary energy method can be used for transverse stresses. In the present paper, only the potential energy method is used.

Stability relations are presented for flat, transversely isotropic plates subjected to in-plane loading and the equations are satisfied for simply supported edges to determine the stability criterion for direct stress loading. Numerical results are presented for isotropic plates in longitudinal compression with the properties of aluminum, which show that, for thicker plates, classical theory gives buckling stresses that are unconservative.

Derivation of Equations of Equilibrium

The strains of nonlinear elasticity (e.g., Ref. 13) are determined for orthogonal curvilinear coordinates. A series expansion is then presented for displacements. For a flat plate, the strains are simplified and the first few terms in the expansion for the displacements are used to obtain nonlinear

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differential equations and boundary conditions appropriate for the analysis of thick homogeneous plates or laminated composite plates. For a simple loading, equations appropriate for buckling are obtained. A derivation similar to that for the flat plate is presented for cylindrical shells.

Strains of Nonlinear Elasticity

Rectangular coordinates (x, y, z) can be represented as functions of orthogonal curvilinear coordinates (u_1, u_2, u_3) and the position vector $\bar{r}$ of a point (x, y, z) in the undeformed elastic body is

$$\bar{r} = ix + jy + kz \tag{1}$$

where (i, j, k) are unit vectors in the rectangular coordinate system. When the elastic body deforms according to the displacement vector

$$\bar{w} = i_1 w_1 + i_2 w_2 + i_3 w_3 \tag{2}$$

where (w_1, w_2, w_3) are functions of (u_1, u_2, u_3). The unit vectors (i_1, i_2, i_3) in the orthogonal curvilinear coordinate system may be considered to be functions of (i, j, k) in addition to (u_1, u_2, u_3). Then the position vector $\bar{r}^*$ of the deformed body is

$$\bar{r}^* = \bar{r} + \bar{w} = ix^* + jy^* + kz^*$$
$$\bar{r}^* = i(x + i \cdot \bar{w}) + j(y + j \cdot \bar{w}) + k(z + k \cdot \bar{w}) \tag{3}$$

The derivative of the undeformed position vector with respect to u_n gives the arc length in the u_n direction,

$$\frac{\partial \bar{r}}{\partial u_n} = i\frac{\partial x}{\partial u_n} + j\frac{\partial y}{\partial u_n} + k\frac{\partial z}{\partial u_n} \tag{4}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}u_n}\right)^2 = \frac{\partial \bar{r}}{\partial u_n} \cdot \frac{\partial \bar{r}}{\partial u_n} = \left(\frac{\partial x}{\partial u_n}\right)^2 + \left(\frac{\partial y}{\partial u_n}\right)^2 + \left(\frac{\partial z}{\partial u_n}\right)^2 \tag{5}$$

where $n = 1,2,3$. Thus, the scale factor in the u_n direction is defined as

$$h_n = \sqrt{\left(\frac{\partial x}{\partial u_n}\right)^2 + \left(\frac{\partial y}{\partial u_n}\right)^2 + \left(\frac{\partial z}{\partial u_n}\right)^2} \tag{6}$$

For the various curvilinear coordinate directions, the scale factors determine the arc length components from the differentials $\mathrm{d}u_n$ through

$$\frac{\mathrm{d}s}{\mathrm{d}u_n} = h_n \tag{7}$$

The derivative of the deformed position vector $\bar{r}^*$ with respect to u_n is

$$\frac{\partial \bar{r}^*}{\partial u_n} = i\frac{\partial x^*}{\partial u_n} + j\frac{\partial y^*}{\partial u_n} + k\frac{\partial z^*}{\partial u_n} \tag{8}$$

so that, in the curvilinear coordinate directions

$$\frac{\mathrm{d}s^*}{\mathrm{d}u_n} = \sqrt{\left(\frac{\partial x^*}{\partial u_n}\right)^2 + \left(\frac{\partial y^*}{\partial u_n}\right)^2 + \left(\frac{\partial z^*}{\partial u_n}\right)^2} \tag{9}$$

The direct strain in the u_n direction is

$$\epsilon_n = \left(\frac{\mathrm{d}s^*}{\mathrm{d}u_n} - \frac{\mathrm{d}s}{\mathrm{d}u_n}\right) \Big/ \frac{\mathrm{d}s}{\mathrm{d}u_n} \qquad \epsilon_n = \frac{1}{h_n}\frac{\mathrm{d}s^*}{\mathrm{d}u_n} - 1 \tag{10}$$

The shear strain γ_{12}, for example, can be found from the triangle formed from a vector joining the ends of the vectors $(1/h_1)(\partial \bar{r}^*/\partial u_1)$ and $(1/h_2)(\partial \bar{r}^*/\partial u_2)$. This new vector is

$$\frac{1}{h_1}\frac{\partial \bar{r}^*}{\partial u_1} - \frac{1}{h_2}\frac{\partial \bar{r}^*}{\partial u_2} \tag{11}$$

which has the magnitude

$$\left[\left(\frac{1}{h_1}\frac{\partial x^*}{\partial u_1} - \frac{1}{h_2}\frac{\partial x^*}{\partial u_2}\right)^2 + \left(\frac{1}{h_1}\frac{\partial y^*}{\partial u_1} - \frac{1}{h_2}\frac{\partial y^*}{\partial u_2}\right)^2 + \left(\frac{1}{h_1}\frac{\partial z^*}{\partial u_1} - \frac{1}{h_2}\frac{\partial z^*}{\partial u_2}\right)^2\right]^{1/2} \tag{12}$$

The cosine of the angle opposite this side of the triangle is equal to the sine of the shearing strain γ_{12}. Therefore, according to the law of cosines

$$\gamma_{12} = \sin^{-1}\left\{\left[\left(\frac{1}{h_1}\frac{\mathrm{d}s^*}{\mathrm{d}u_1}\right)^2 + \left(\frac{1}{h_2}\frac{\mathrm{d}s^*}{\mathrm{d}u_2}\right)^2 - \left|\frac{1}{h_1}\frac{\partial \bar{r}^*}{\partial u_1} - \frac{1}{h_2}\frac{\partial \bar{r}^*}{\partial u_2}\right|^2\right] \Big/ \left(2\frac{1}{h_1}\frac{\mathrm{d}s^*}{\mathrm{d}u_1}\frac{1}{h_2}\frac{\mathrm{d}s^*}{\mathrm{d}u_2}\right)\right\} \tag{13}$$

and similarly for the other shearing strains.

Flat Plate Strains

For the special case of the flat plate

$$u_1 = x, \quad u_2 = y, \quad u_3 = z, \quad h_x = h_y = h_z = 1 \tag{14}$$

then with

$$\bar{w} = iu + jv + kw \tag{15}$$

the resulting strains are

$$\epsilon_x = \sqrt{(1 + u_{,x})^2 + v_{,x}^2 + w_{,x}^2} - 1$$
$$\epsilon_y = \sqrt{u_{,y}^2 + (1 + v_{,y})^2 + w_{,y}^2} - 1$$
$$\epsilon_z = \sqrt{u_{,z}^2 + v_{,z}^2 + (1 + w_{,z})^2} - 1 \tag{16}$$

$$\gamma_{xy} = \sin^{-1}\left[\frac{(1 + u_{,x})u_{,y} + v_{,x}(1 + v_{,y}) + w_{,x}w_{,y}}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2 + w_{,x}^2}\sqrt{u_{,y}^2 + (1 + v_{,y})^2 + w_{,y}^2}}\right] \tag{17}$$

and similarly for γ_{xz} and γ_{yz}. The notation $u_{,x} = \partial u/\partial x$, etc., is used.

Circular Cylinder Strains

For the special case of the circular cylinder

$$u_1 = x, \quad u_2 = \theta, \quad u_3 = r$$
$$h_x = 1, \quad h_\theta = r, \quad h_r = 1 \tag{18}$$

then with

$$\bar{w} = i_x u + i_\theta v + i_r w \tag{19}$$

and

$$i_x = i, \quad i_\theta = j\cos\theta - k\sin\theta$$
$$i_r = j\sin\theta + k\cos\theta \tag{20}$$

The strains are found to be

$$\epsilon_x = \sqrt{(1+u_{,x})^2 + v_{,x}^2 + w_{,x}^2} - 1$$

$$\epsilon_\theta = \sqrt{\left(\frac{u_{,\theta}}{r}\right)^2 + \left(1 + \frac{v_{,\theta}}{r} + \frac{w}{r}\right)^2 + \left(\frac{w_{,\theta}}{r} - \frac{v}{r}\right)^2} - 1$$

$$\epsilon_r = \sqrt{u_{,r}^2 + v_{,r}^2 + (1+w_{,r})^2} - 1$$

$$\gamma_{x\theta} = \sin^{-1}\left\{\left[(1+u_{,x})\frac{u_{,\theta}}{r} + v_{,x}\left(1+\frac{v_{,\theta}}{r}+\frac{w}{r}\right) + w_{,x}\left(\frac{w_{,\theta}}{r} - \frac{v}{r}\right)\right] \Bigg/ \left[\sqrt{(1+u_{,x})^2 + v_{,x}^2 + w_{,x}^2} \times \sqrt{\frac{u_{,\theta}}{r} + \left(1+\frac{v_{,\theta}}{r}+\frac{w}{r}\right)^2 + \left(\frac{w_{,\theta}}{r} - \frac{v}{r}\right)^2}\,\right]\right\} \tag{21}$$

and similarly for γ_{xr} and $\gamma_{\theta r}$. The strains that have been derived are the exact strains of nonlinear elasticity. Approximate strains based on the expansion of these exact strains starting from linear values will be used in the present paper.

Series Expansion for Displacements

The series for w_n, in terms of the coordinate normal to the midsurface u_3, is taken to be

$$w_n = w_n^0 + w_n^a\frac{u_3}{h} + w_n^{2a}\left(\frac{u_3}{h}\right)^2 + \sum_{m=1,3}^{\infty}\left(w_n^{ms}\sin\frac{m\pi u_3}{h} + w_n^{mc}\cos\frac{m\pi u_3}{h}\right) \qquad n=1,2,3 \tag{22}$$

The coefficients on the right-hand side of the equation shown (e.g., w_n^0, w_n^a, etc.) are functions of u_1 and u_2. If all the trigonometric terms were used, there would be no need for the three polynomial terms in w_n, since the trigonometric terms form a complete set. Term by term, the symmetric and the antisymmetric parts of the trigonometric set satisfy certain conditions at the shell surfaces $u_3 = \pm h/2$. If the complete set is not used, then the other three terms, or at least some of them, are desirable to have because they permit any desired behavior near the shell surfaces. In the equations that follow, only a few trigonometric terms are retained.

Nonlinear Equations for the Flat Plate

The strains given by Eqs. (16) and (17) are expanded into a Taylor series in which all first- and second-degree terms are retained. This series yields the following approximate strains for the flat plate:

$$\epsilon_x = u_{,x} + \tfrac{1}{2}v_{,x}^2 + \tfrac{1}{2}w_{,x}^2$$

$$\epsilon_y = v_{,y} + \tfrac{1}{2}u_{,y}^2 + \tfrac{1}{2}w_{,y}^2$$

$$\epsilon_z = w_{,z} + \tfrac{1}{2}u_{,z}^2 + \tfrac{1}{2}v_{,z}^2$$

$$\gamma_{xy} = u_{,y}(1 - v_{,y}) + v_{,x}(1 - u_{,x}) + w_{,x}w_{,y}$$

$$\gamma_{xz} = w_{,x}(1 - u_{,x}) + u_{,z}(1 - w_{,z}) + v_{,x}v_{,z}$$

$$\gamma_{yz} = v_{,z}(1 - w_{,z}) + w_{,y}(1 - v_{,y}) + u_{,y}u_{,z} \tag{23}$$

The first few terms of the series expansion for the displacements u, v, w for which equations are derived are

$$u = u^0 + u^a\frac{z}{h} + u^s\sin\frac{\pi z}{h} \qquad v = v^0 + v^a\frac{z}{h} + v^s\sin\frac{\pi z}{h}$$

$$w = w^0 + w^c\cos\frac{\pi z}{h} \tag{24}$$

The results presented in Ref. 12 show that additional terms are not needed for the range of applications considered. In conventional transverse shearing plate theory (also identified as sandwich plate theory), the trigonometric terms are absent, and the rotations u^a/h and v^a/h are usually identified as β_x and β_y, respectively. In classical plate theory, the trigonometric terms are also absent, and u^a/h and v^a/h are replaced by $-w_{,x}^0$, and $-w_{,y}^0$, respectively.

If attention is restricted to problems where the in-plane displacements are small and the largest rotations are $w_{,x}^0$ and $w_{,y}^0$, then products not containing these rotations can be neglected.

With this assumption the strains become

$$\epsilon_x = u_{,x}^0 + \frac{1}{2}w_{,x}^{02} + u_{,x}^a\frac{z}{h} + u_{,x}^s\sin\frac{\pi z}{h}$$

$$\epsilon_y = v_{,y}^0 + \frac{1}{2}w_{,y}^{02} + v_{,y}^a\frac{z}{h} + v_{,y}^s\sin\frac{\pi z}{h}$$

$$\epsilon_z = -\frac{\pi}{h}w^c\sin\frac{\pi z}{h}$$

$$\gamma_{xy} = u_{,y}^0 + v_{,x}^0 + w_{,x}^0 w_{,y}^0 + (u_{,y}^a + v_{,x}^a)\frac{z}{h} + (u_{,y}^s + v_{,x}^s)\sin\frac{\pi z}{h}$$

$$\gamma_{xz} = w_{,x}^0 + \frac{u^a}{h} + \left(w_{,x}^c + \frac{\pi}{h}u^s\right)\cos\frac{\pi z}{h}$$

$$\gamma_{yz} = \frac{v^a}{h} + w_{,y}^0 + \left(\frac{\pi}{h}v^s + w_{,y}^c\right)\cos\frac{\pi z}{h} \tag{25}$$

Most plate problems do have small in-plane deformations, but a case where this assumption might not be valid for determining the behavior is a stiffener web for a stiffened plate that may have large in-plane deformations. The virtual work or virtual potential energy of the internal forces for a three-dimensional body is

$$\delta\Pi = \int_0^a\int_0^b\int_{-h/2}^{h/2}(\sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + \sigma_z\delta\epsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz} + \tau_{xy}\delta\gamma_{xy})\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \tag{26}$$

In the virtual work of the plate, the volume is assumed constant under deformation. If external forces are applied, their virtual work must be added. Define

$$\begin{aligned}
N_x &= \int\sigma_x\,\mathrm{d}z & M_{xy} &= \int\tau_{xy}z\,\mathrm{d}z \\
N_y &= \int\sigma_y\,\mathrm{d}z & \mathcal{P}_z &= \frac{\pi}{h}\int\sigma_z\sin\frac{\pi z}{h}\,\mathrm{d}z \\
N_{yz} &= \int\tau_{yz}\,\mathrm{d}z & \mathcal{N}_{yz} &= \int\tau_{yz}\cos\frac{\pi z}{h}\,\mathrm{d}z \\
N_{xz} &= \int\tau_{xz}\,\mathrm{d}z & \mathcal{N}_{xz} &= \int\tau_{xz}\cos\frac{\pi z}{h}\,\mathrm{d}z \\
N_{xy} &= \int\tau_{xy}\,\mathrm{d}z & \mathcal{M}_x &= \frac{h}{\pi}\int\sigma_x\sin\frac{\pi z}{h}\,\mathrm{d}z \\
M_x &= \int\sigma_x z\,\mathrm{d}z & \mathcal{M}_y &= \frac{h}{\pi}\int\sigma_y\sin\frac{\pi z}{h}\,\mathrm{d}z \\
M_y &= \int\sigma_y z\,\mathrm{d}z & \mathcal{M}_{xy} &= \frac{h}{\pi}\int\tau_{xy}\sin\frac{\pi z}{h}\,\mathrm{d}z
\end{aligned} \tag{27}$$

where the limits on the integrals above are from $-h/2$ to $h/2$. Integration by parts leads to the following equations of equilibrium:

$$N_{x,x} + N_{xy,y} = 0$$
$$N_{y,y} + N_{xy,x} = 0$$
$$N_{xz,x} + N_{yz,y} + \left(N_x w^0_{,x}\right)_{,x} + \left(N_y w^0_{,y}\right)_{,y} + \left(N_{xy} w^0_{,x}\right)_{,y} + \left(N_{xy} w^0_{,y}\right)_{,x} = 0$$
$$M_{x,x} + M_{xy,y} - N_{xz} = 0$$
$$M_{y,y} + M_{xy,x} - N_{yz} = 0$$
$$\mathscr{M}_{x,x} + \mathscr{M}_{xy,y} - \mathscr{N}_{xz} = 0$$
$$\mathscr{M}_{y,y} + \mathscr{M}_{xy,x} - \mathscr{N}_{yz} = 0$$
$$\mathscr{N}_{xz,x} + \mathscr{N}_{yz,y} + \mathscr{P}_z = 0 \qquad (28)$$

and the variationally consistent boundary conditions

at $x = 0, a$:	at $y = 0, b$:
$N_x u^0 = 0$	$N_{xy} u^0 = 0$
$N_{xy} v^0 = 0$	$N_y v^0 = 0$
$(N_{xz} + N_x w^0_{,x} + N_{xy} w^0_{,y}) w^0 = 0$	$(N_{yz} + N_y w^0_{,y} + N_{xy} w^0_{,x}) w^0 = 0$
$M_x u^a = 0$	$M_{xy} u^a = 0$
$M_{xy} v^a = 0$	$M_y v^a = 0$
$\mathscr{M}_x u^s = 0$	$\mathscr{M}_{xy} u^s = 0$
$\mathscr{M}_{xy} v^s = 0$	$\mathscr{M}_y v^s = 0$
$\mathscr{N}_{xz} w^c = 0$	$\mathscr{N}_{yz} w^c = 0$ (29)

According to Hooke's law, the stresses are related to the strains by

$$\begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} = \left[C_{ij}\right] \begin{Bmatrix} \epsilon \\ \gamma \end{Bmatrix} \qquad (30)$$

where the strains are those associated with the assumed displacements. For a laminated filamentary composite, the C_{ij} are the same set of constants for each lamina having the same orientation of filaments, but are different for other orientations. In integrating over z, as required here, the C_{ij} may change from layer to layer and, hence, must be treated as functions of z.

Stability Relations for Transversely Isotropic Plates

Consider a class of isotropic plates and transversely isotropic laminated plates with

$$\left[C_{ij}\right] = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \qquad (31)$$

for each layer and subjected to uniform in-plane loading for which $w^0 = 0$ prior to buckling. Stability equations may be set up by using the adjacent equilibrium criterion (see Ref. 14). For a plate, the first and second equations of Eqs. (28) determine the values $N_x = -N_x^b$, $N_y = -N_y^b$, $N_{xy} = N_{xy}^b$, which are taken to be constants representing the applied forces or displacements prior to buckling. The criterion determines the stability equations to be

$$N_{xz,x} + N_{yz,y} - N_x^b w^0_{,xx} - N_y^b w^0_{,yy} + 2N_{xy}^b w^0_{,xy} = 0$$
$$M_{x,x} + M_{xy,y} - N_{xz} = 0$$
$$M_{y,y} + M_{xy,x} - N_{yz} = 0$$
$$\mathscr{M}_{x,x} + \mathscr{M}_{xy,y} - \mathscr{N}_{xz} = 0$$
$$\mathscr{M}_{y,y} + \mathscr{M}_{xy,x} - \mathscr{N}_{yz} = 0$$
$$\mathscr{N}_{xz,x} + \mathscr{N}_{yz,y} + \mathscr{P}_z = 0 \qquad (32)$$

where

$$N_{yz} = A_{44}\left(w^0_{,y} + \frac{v^a}{h}\right) + H_{44}\left(w^c_{,y} + \frac{\pi}{h} v^s\right)$$
$$N_{xz} = A_{55}\left(w^0_{,x} + \frac{u^a}{h}\right) + H_{55}\left(w^c_{,x} + \frac{\pi}{h} u^s\right)$$
$$M_x = \frac{1}{h}\left(D_{11} u^a_{,x} + D_{12} v^a_{,y} + \pi O_{11} u^s_{,x} + \pi O_{12} v^s_{,y}\right)$$
$$M_y = \frac{1}{h}\left(D_{22} v^a_{,y} + D_{12} u^a_{,x} + \pi O_{22} v^s_{,y} + \pi O_{12} u^s_{,x}\right)$$
$$M_{xy} = \frac{1}{h}\left[D_{66}\left(u^a_{,y} + v^a_{,x}\right) + \pi O_{66}\left(u^s_{,y} + v^s_{,x}\right)\right]$$
$$\mathscr{P}_z = -\mathscr{z}_{33} w^c$$
$$\mathscr{N}_{yz} = H_{44}\left(w^0_{,y} + \frac{v^a}{h}\right) + \mathscr{A}_{44}\left(w^c_{,y} + \frac{\pi}{h} v^s\right)$$
$$\mathscr{N}_{xz} = H_{55}\left(w^0_{,x} + \frac{u^a}{h}\right) + \mathscr{A}_{55}\left(w^c_{,x} + \frac{\pi}{h} u^s\right)$$
$$\mathscr{M}_x = \frac{1}{h}\left(O_{11} u^a_{,x} + O_{12} v^a_{,y} + \pi \mathscr{D}_{11} u^s_{,x} + \pi \mathscr{D}_{12} v^s_{,y}\right)$$
$$\mathscr{M}_y = \frac{1}{h}\left(O_{22} v^a_{,y} + O_{12} u^a_{,x} + \pi \mathscr{D}_{22} v^s_{,y} + \pi \mathscr{D}_{12} u^s_{,x}\right)$$
$$\mathscr{M}_{xy} = \frac{1}{h}\left[O_{66}\left(u^a_{,y} + v^a_{,x}\right) + \pi \mathscr{D}_{66}\left(u^s_{,y} + v^s_{,x}\right)\right] \qquad (33)$$

and

$$A_{ij} = \int C_{ij}\,\mathrm{d}z, \quad H_{ij} = \int C_{ij} \cos\frac{\pi z}{h}\,\mathrm{d}z$$
$$\mathscr{A}_{ij} = \int C_{ij} \cos^2\frac{\pi z}{h}\,\mathrm{d}z, \quad D_{ij} = \int C_{ij} z^2\,\mathrm{d}z$$
$$O_{ij} = \frac{h}{\pi}\int C_{ij} z \sin\frac{\pi z}{h}\,\mathrm{d}z, \quad \mathscr{D}_{ij} = \left(\frac{h}{\pi}\right)^2 \int C_{ij} \sin^2\frac{\pi z}{h}\,\mathrm{d}z$$
$$\mathscr{z}_{ij} = \left(\frac{\pi}{h}\right)^2 \int C_{ij} \sin^2\frac{\pi z}{h}\,\mathrm{d}z \qquad (34)$$

where again the limits on the integrals above are from $-h/2$ to $h/2$.

According to the adjacent equilibrium criterion, all of the forces (except the superscript b forces) and displacements in the stability equations are redefined as infinitesimal initial changes that occur due to buckling.

Solution of the Stability Equations for Direct Stress Loading of a Simply Supported Rectangular Orthotropic Plate

The solution of the stability equations with $N_{xy}^b = 0$ can be obtained for a rectangular plate with simply supported edges such that

$$\begin{array}{ll} \text{at } x=0,a: & \text{at } y=0,b: \\ w^0 = w^c = 0 & w^0 = w^c = 0 \\ v^a = v^s = 0 & u^a = u^s = 0 \\ M_x = \mathcal{M}_x = 0 & M_y = \mathcal{M}_y = 0 \end{array} \tag{35}$$

The differential equations and boundary conditions are satisfied by

$$\frac{u^a}{h} = U\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \frac{\pi u^s}{h} = \mathcal{U}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{v^a}{h} = V\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad \frac{\pi v^s}{h} = \mathcal{V}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$w^0 = W\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad w^c = \mathcal{W}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$m = 1,2,\ldots,\ n = 1,2,\ldots \tag{36}$$

Upon substitution of Eqs. (36), the stability equations (32) become

$$[M_{ij}]\begin{vmatrix} U \\ V \\ W \\ \mathcal{U} \\ \mathcal{V} \\ \mathcal{W} \end{vmatrix} = 0 \tag{37}$$

where

$$M_{11} = A_{55} + \left(\frac{m\pi}{a}\right)^2 D_{11} + \left(\frac{n\pi}{b}\right)^2 D_{66}$$

$$M_{12} = \frac{m\pi}{a}\frac{n\pi}{b}(D_{12} + D_{66})$$

$$M_{13} = \frac{m\pi}{a}A_{55}$$

$$M_{14} = H_{55} + \left(\frac{m\pi}{a}\right)^2 O_{11} + \left(\frac{n\pi}{b}\right)^2 O_{66}$$

$$M_{15} = \frac{m\pi}{a}\frac{n\pi}{b}(O_{12} + O_{66}) = M_{24}$$

$$M_{16} = \frac{m\pi}{a}H_{55} = M_{34}$$

$$M_{22} = A_{44} + \left(\frac{n\pi}{b}\right)^2 D_{22} + \left(\frac{m\pi}{a}\right)^2 D_{66}$$

$$M_{23} = \frac{n\pi}{b}A_{44} = M_{35}$$

$$M_{25} = H_{44} + \left(\frac{n\pi}{b}\right)^2 O_{22} + \left(\frac{m\pi}{a}\right)^2 O_{66}$$

$$M_{26} = \frac{n\pi}{b}H_{44}$$

$$M_{33} = \left(\frac{m\pi}{a}\right)^2 A_{55} + \left(\frac{n\pi}{b}\right)^2 A_{44} - N_x^b\left(\frac{m\pi}{a}\right)^2 - N_y^b\left(\frac{n\pi}{b}\right)^2$$

$$M_{36} = \left(\frac{m\pi}{a}\right)^2 H_{55} + \left(\frac{n\pi}{b}\right)^2 H_{44}$$

$$M_{44} = \mathcal{A}_{55} + \left(\frac{m\pi}{a}\right)^2 \mathcal{D}_{11} + \left(\frac{n\pi}{b}\right)^2 \mathcal{D}_{66}$$

$$M_{45} = \frac{m\pi}{a}\frac{n\pi}{b}(\mathcal{D}_{12} + \mathcal{D}_{66})$$

$$M_{46} = \frac{m\pi}{a}\mathcal{A}_{55}$$

$$M_{55} = \mathcal{A}_{44} + \left(\frac{n\pi}{b}\right)^2 \mathcal{D}_{22} + \left(\frac{m\pi}{a}\right)^2 \mathcal{D}_{66}$$

$$M_{56} = \frac{n\pi}{b}\mathcal{A}_{44}$$

$$M_{66} = \left(\frac{m\pi}{a}\right)^2 \mathcal{A}_{55} + \left(\frac{n\pi}{b}\right)^2 \mathcal{A}_{44} + z_{33}$$

and since matrix M is symmetric, $M_{ji} = M_{ij}$. The determinant of the (square) matrix set equal to zero is the stability criterion. If the trigonometric terms $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are set equal to zero, the results so obtained can be identified as those from the conventional transverse shear theory. Further reduction to classical theory may be done through limiting processes; however, such reduction is more involved. If all of the terms are retained, all of the stiffnesses in three-dimensional elasticity are represented. The model is then more flexible than that resulting from classical theory or from conventional transverse shear theory.

A numerical example is presented for longitudinally compressed isotropic aluminum plates. The stiffness matrix considered is

$$[C_{ij}] = \begin{bmatrix} E/(1-\mu^2) & \mu E/(1-\mu^2) & 0 & 0 & 0 & 0 \\ \mu E/(1-\mu^2) & E/(1-\mu^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{38}$$

where $E = 10.7 \times 10^6$ psi, $\mu = 0.33$, and $G = E/[2(1+\mu)]$.

Buckling results for a simply supported aluminum plate are presented in Fig. 1, which shows the variation in the buckling stress coefficient with the thickness ratio as given by the three theories (classical, conventional transverse shearing, and three-dimensional flexibility) for a range of length-to-width aspect ratios. These results show that, for aluminum plates with thickness greater than 10% of the width of the plate, the effects of transverse shearing should be included in determining the compressive buckling stress. This lower buckling stress is important in determining the buckling load of the crown of a hat stiffener in a stiffened panel in bending or compression and the start of postbuckling.

Nonlinear Equations for a Circular Cylindrical Shell

The nonlinear equations of equilibrium for a cylindrical shell are derived in a manner similar to that used for the flat plate. The strains for a cylindrical shell, retaining terms up to

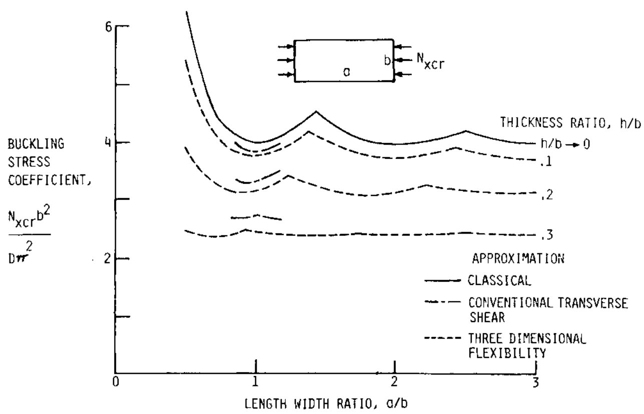

Fig. 1 Variation of buckling stress coefficient of simply supported aluminum plates with thickness *h*.

second degree, are

$$\epsilon_x = u_{,x} + \tfrac{1}{2}v_{,x}^2 + \tfrac{1}{2}w_{,x}^2$$

$$\epsilon_\theta = \frac{v_{,\theta}}{r} + \frac{w}{r} + \frac{1}{2}\left(\frac{u_{,\theta}}{r}\right)^2 + \frac{1}{2}\left(\frac{w_{,\theta}}{r} - \frac{v}{r}\right)^2$$

$$\epsilon_r = w_{,r} + \tfrac{1}{2}u_{,r}^2 + \tfrac{1}{2}v_{,r}^2$$

$$\gamma_{x\theta} = \frac{u_{,\theta}}{r}\left(1 - \frac{v_{,\theta}}{r} - \frac{w}{r}\right) + v_{,x}(1 - u_{,x}) + w_{,x}\left(\frac{w_{,\theta}}{r} - \frac{v}{r}\right)$$

$$\gamma_{xr} = w_{,x}(1 - u_{,x}) + u_{,r}(1 - w_{,r}) + v_{,x}v_{,r}$$

$$\gamma_{\theta r} = v_{,r}(1 - w_{,r}) + \left(\frac{w_{,\theta}}{r} - \frac{v}{r}\right)\left(1 - \frac{v_{,\theta}}{r} - \frac{w}{r}\right) + \frac{u_{,\theta}}{r}u_{,r} \tag{39}$$

The first few terms in the series expansion for the displacements (u, v, w) that are retained for a cylindrical shell of radius R are

$$u = u^0 + u^a\frac{r-R}{h} + u^s\sin\frac{\pi(r-R)}{h}$$

$$v = v^0 + v^a\frac{r-R}{h} + v^s\sin\frac{\pi(r-R)}{h}$$

$$w = w^0 + w^a\frac{r-R}{h} + w^c\cos\frac{\pi(r-R)}{h} \tag{40}$$

The term w^a is included here, but, not in the plate theory, since it appears in the linear parts of an in-plane strain in much the same way as u^a and v^a.

If attention is restricted to problems where the largest rotations are $w^0_{,x}$ and $(w^0_{,\theta} - v^0)/r$ so that products not containing these terms can be neglected, the strains become

$$\epsilon_x = u^0_{,x} + \tfrac{1}{2}w^{02}_{,x} + u^a_{,x}\frac{r-R}{h} + u^s_{,x}\sin\frac{\pi(r-R)}{h}$$

$$\epsilon_\theta = \frac{v^0_{,\theta} + w^0}{r} + \frac{\tfrac{1}{2}(w^0_{,\theta} - v^0)^2}{r^2} + \left(\frac{v^a_{,\theta} + w^a}{r}\right)\left(\frac{r-R}{h}\right) + \left(\frac{v^s_{,\theta}}{r}\right)\sin\frac{\pi(r-R)}{h} + \left(\frac{w^c}{r}\right)\cos\frac{\pi(r-R)}{h}$$

$$\epsilon_r = \frac{w^a}{h} - \frac{\pi}{h}w^c\sin\frac{\pi(r-R)}{h}$$

$$\gamma_{x\theta} = \frac{u^0_{,\theta}}{r} + v^0_{,x} + \frac{w^0_{,x}(w^0_{,\theta} - v^0)}{r} + \left(\frac{u^a_{,\theta}}{r} + v^a_{,x}\right)\frac{r-R}{h} + \left(\frac{u^s_{,\theta}}{r} + v^s_{,x}\right)\sin\frac{\pi(r-R)}{h}$$

$$\gamma_{xr} = w^0_{,x} + \frac{u^a}{h} + w^a_{,x}\frac{r-R}{h} + \left(w^c_{,x} + \frac{\pi}{h}u^s\right)\cos\frac{\pi(r-R)}{h}$$

$$\gamma_{\theta r} = \frac{w^0_{,\theta} - v^0}{r} + \frac{v^a}{h} + \left(\frac{w^a_{,\theta} - v^a}{r}\right)\frac{r-R}{h} + \left(\frac{w^c_{,\theta}}{r} + \frac{\pi}{h}v^s\right)\cos\frac{\pi(r-R)}{h} - \left(\frac{v^s}{r}\right)\sin\frac{\pi(r-R)}{h} \tag{41}$$

The three-dimensional virtual work of a shell with surface pressures $p_{h/2}$ and $p_{-h/2}$ is

$$\delta\pi = \int_0^a\int_{\theta_0}^{\theta_b}\int_{R-h/2}^{R+h/2}(\sigma_x\delta\epsilon_x + \sigma_\theta\delta\epsilon_\theta + \sigma_r\delta\epsilon_r + \tau_{x\theta}\delta\gamma_{x\theta} + \tau_{xr}\delta\gamma_{xr} + \tau_{\theta r}\delta\gamma_{\theta r})\,r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x$$
$$+ \int_0^a\int_{\theta_0}^{\theta_b}p_{h/2}\delta w\left(x,\theta,R+\frac{h}{2}\right)\left(R+\frac{h}{2}\right)\mathrm{d}\theta\,\mathrm{d}x + \int_0^a\int_{\theta_0}^{\theta_b}p_{-h/2}\delta w\left(x,\theta,R-\frac{h}{2}\right)\left(R-\frac{h}{2}\right)\mathrm{d}\theta\,\mathrm{d}x \tag{42}$$

With the strains listed above, integration by parts leads to the following equations of equilibrium:

$$\int_{R-(h/2)}^{R+(h/2)}\left(\sigma_{x,x} + \frac{\tau_{x\theta,\theta}}{r}\right)r\,\mathrm{d}r = 0$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\frac{\sigma_{\theta,\theta}}{r} + \frac{\sigma_\theta(w^0_{,\theta} - v^0)}{r^2} + \tau_{x\theta,x} + \frac{\tau_{x\theta}w^0_{,x}}{r} + \frac{\tau_{\theta r}}{r}\right]r\,\mathrm{d}r = 0$$

$$\int_{R-(h/2)}^{R+(h/2)}\left\{(\sigma_x w^0_{,x})_{,x} - \frac{\sigma_\theta}{r} + \frac{[\sigma_\theta(w^0_{,\theta} - v^0)]_{,\theta}}{r^2} + \frac{[\tau_{x\theta}(w^0_{,\theta} - v^0)]_{,x}}{r} + \frac{(\tau_{x\theta}w^0_{,x})_{,\theta}}{r} + \tau_{xr,x} + \frac{\tau_{\theta r,\theta}}{r}\right\}r\,\mathrm{d}r = pR$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\left(\sigma_{x,x} + \frac{\tau_{x\theta,\theta}}{r}\right)\frac{r-R}{h} - \frac{\tau_{xr}}{h}\right]r\,\mathrm{d}r = 0$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\left(\frac{\sigma_{\theta,\theta}}{r} + \tau_{x\theta,x} + \frac{\tau_{\theta r}}{r}\right)\frac{r-R}{h} - \frac{\tau_{\theta r}}{h}\right]r\,\mathrm{d}r = 0$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\left(-\frac{\sigma_\theta}{R} + \tau_{xr,x} + \frac{\tau_{\theta r,\theta}}{r}\right)\frac{r-R}{h} - \frac{\sigma_r}{h}\right]r\,\mathrm{d}r = ph/4$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\left(\sigma_{x,x} + \frac{\tau_{x\theta,\theta}}{r}\right)\sin\frac{\pi(r-R)}{h} - \tau_{xr}\frac{\pi}{h}\cos\frac{\pi(r-R)}{h}\right]r\,\mathrm{d}r = 0$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\left(\frac{\sigma_{\theta,\theta}}{r} + \tau_{x\theta,x} + \frac{\tau_{\theta r}}{r}\right)\sin\frac{\pi(r-R)}{h} - \tau_{\theta r}\frac{\pi}{h}\cos\frac{\pi(r-R)}{h}\right]r\,\mathrm{d}r = 0$$

$$\int_{R-(h/2)}^{R+(h/2)}\left[\left(-\frac{\sigma_\theta}{r} + \tau_{xr,x} + \frac{\tau_{\theta r,\theta}}{r}\right)\cos\frac{\pi(r-R)}{h} + \sigma_r\frac{\pi}{h}\sin\frac{\pi(r-R)}{h}\right]r\,\mathrm{d}r = 0 \tag{43}$$

If r is replaced by $R + z$, then the above equations become

$$N_{x,x} + \frac{N_{x\theta,\theta}}{R} + \frac{M_{x,x}}{R} = 0$$

$$\frac{N_{\theta,\theta}}{R} + N_{x\theta,x} + \frac{N_{\theta z}}{R} + \frac{M_{x\theta,x}}{R} + \frac{P_\theta(w^0_{,\theta} - v^0)}{R} + \frac{N_{x\theta}w^0_{,x}}{R} = 0$$

$$N_{xz,x} + \frac{N_{\theta z,\theta}}{R} - \frac{N_\theta}{R} + \frac{M_{xz,x}}{R} + (N_x w^0_{,x})_{,x} + \frac{[P_\theta(w^0_{,\theta} - v^0)]_{,\theta}}{R} + \frac{[N_{x\theta}(w^0_{,\theta} - v^0)]_{,x}}{R} + \frac{(N_{x\theta}w^0_{,x})_{,\theta}}{R} + \frac{(M_x w^0_{,x})_{,x}}{R} = p$$

$$M_{x,x} + \frac{M_{x\theta,\theta}}{R} - N_{xz} + \frac{K_{x,x}}{R} - \frac{M_{xz}}{R} = 0$$

$$\frac{M_{\theta,\theta}}{R} + M_{x\theta,x} - N_{\theta z} + \frac{K_{x\theta,x}}{R} = 0$$

$$\mathcal{M}_{x,x} + \frac{\mathcal{M}_{x\theta,\theta}}{R} - \mathcal{N}_{xz} + \frac{\mathcal{K}_{x,x}}{R} - \frac{\mathcal{L}_{xz}}{R} = 0$$

$$\frac{\mathcal{M}_{\theta,\theta}}{R} + \mathcal{M}_{x\theta,x} - \mathcal{N}_{\theta z} + \frac{\mathcal{K}_{x\theta,x}}{R} + \frac{M_{\theta z}}{R} - \frac{\mathcal{L}_{\theta z}}{R} = 0$$

$$\mathcal{N}_{xz,x} + \frac{\mathcal{N}_{\theta z,\theta}}{R} + \mathcal{P}_z - \frac{\mathcal{N}_\theta}{R} + \frac{\mathcal{L}_{xz,x}}{R} + \frac{\mathcal{Q}_z}{R} = 0 \tag{44}$$

where the N, M, $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{P}_z$ are similar to those used for the flat plate in Eqs. (27) and

$$P_\theta = \int \frac{\sigma_\theta}{r+z}\,\mathrm{d}z \qquad \mathscr{K}_x = \frac{h}{\pi}\int \sigma_x z \sin\frac{\pi z}{h}\,\mathrm{d}z$$

$$K_x = \int \sigma_x z^2\,\mathrm{d}z \qquad \mathscr{K}_{x\theta} = \frac{h}{\pi}\int \tau_{x\theta} z \sin\frac{\pi z}{h}\,\mathrm{d}z$$

$$K_{x\theta} = \int \tau_{x\theta} z^2\,\mathrm{d}z \qquad \mathscr{L}_{xz} = \int \tau_{xr} z \cos\frac{\pi z}{h}\,\mathrm{d}z$$

$$K_{xz} = \int \tau_{xr} z^2\,\mathrm{d}z \qquad \mathscr{L}_{\theta z} = \int \tau_{\theta r} z \cos\frac{\pi z}{h}\,\mathrm{d}z$$

$$\mathscr{Q}_z = \frac{\pi}{h}\int \sigma_r z \sin\frac{\pi z}{h}\,\mathrm{d}z \tag{45}$$

and the limits on the integrals above are from $-h/2$ and $h/2$ and

$$p = p_{h/2} + p_{-h/2}$$
$$p_{h/2} = p_{-h/2}$$

The variationally consistent boundary conditions are

at $x = 0, a$: | at $\theta = \theta_0, \theta_b$:

$$(N_x + M_x/R)u^0 = 0 \qquad N_{x\theta}u^0 = 0$$
$$(N_{x\theta} + M_{x\theta}/R)v^0 = 0 \qquad N_\theta v^0 = 0$$
$$\left[N_{xz} + \frac{M_{xz}}{R} + N_x w^0_{,x} + \frac{N_{x\theta}(w^0_{,\theta} - v^0)}{R} + \frac{M_x w^0_{,x}}{R}\right] w^0 = 0 \qquad \left[N_{\theta z} + \frac{P_\theta(w^0_{,\theta} - v^0)}{R} + N_{x\theta}w^0_{,x}\right] w^0 = 0$$
$$(M_x + K_x/R)u^a \qquad M_{x\theta}u^a = 0$$
$$(M_{x\theta} + K_{x\theta}/R)v^a \qquad M_\theta v^a = 0$$
$$(\mathscr{M}_x + K_x/R)u^s \qquad \mathscr{M}_{x\theta}u^s = 0$$
$$(\mathscr{M}_{x\theta} + \mathscr{K}_{x\theta}/R)v^s \qquad \mathscr{M}_\theta v^s = 0$$
$$(\mathscr{N}_{xz} + \mathscr{K}_{x\theta}/R)w^c \qquad \mathscr{N}_{\theta z}w^c = 0 \tag{46}$$

The stresses are related to the strains according to Hooke's law [Eq. (29)], where the strains are those associated with the assumed displacements.

Equations for a Shallow Cylindrical Shell

This section will give equations for the thick shell that are analogous to the Donnell-Mushtari-Vlasov equations. These equations are generally simpler to work with than ones that do not make use of the shallow shell assumption, but they are not valid for loading conditions that generate deformation modes with two or three waves around the circumference or when the shell has a radius that is not much larger than the thickness. To obtain the shallow cylindrical shell equations, z is used in the radial direction instead of r such that $r = R + z$, where R is the constant radius and z is neglected compared to R in the sense that h is small compared to R. Also, y is used instead of θ such that $\mathrm{d}y = R\,\mathrm{d}\theta$. For a shallow cylinder the strains [Eqs. (41)] become

$$\epsilon_x = u^0_{,x} + \frac{1}{2}w^{02}_{,x} + u^a_{,x}\frac{z}{h} + u^s_{,x}\sin\frac{\pi z}{h}$$

$$\epsilon_y = v^0_{,y} + \frac{w^0}{R} + \frac{1}{2}\left(w^0_{,y} - \frac{v^0}{R}\right)^2 + \left(v^a_{,y} + \frac{w^a}{R}\right)\frac{z}{h} + v^s_{,y}\sin\frac{\pi z}{h} + \frac{w^c}{R}\cos\frac{\pi z}{h}$$

$$\epsilon_z = \frac{w^a}{h} - \frac{\pi}{h}w^c\sin\frac{\pi z}{h}$$

$$\gamma_{xy} = u^0_{,y} + v^0_{,x} + w^0_{,x}\left(w^0_{,y} - \frac{v^0}{R}\right) + \left(u^a_{,y} + v^a_{,x}\right)\frac{z}{h} + \left(u^s_{,y} + v^s_{,x}\right)\sin\frac{\pi z}{h}$$

$$\gamma_{xz} = w^0_{,x} + \frac{u^a}{h} + w^a_{,x}\frac{z}{h} + \left(w^c_{,x} + \frac{\pi}{h}u^s\right)\cos\frac{\pi z}{h}$$

$$\gamma_{yz} = w^0_{,y} - \frac{v^0}{R} + \frac{v^a}{h} + \left(w^a_{,y} - \frac{v^a}{R}\right)\frac{z}{h} - \frac{v^s}{R}\sin\frac{\pi z}{h} + \left(w^c_{,y} + \frac{\pi}{h}v^s\right)\cos\frac{\pi z}{h} \tag{47}$$

In conventional transverse shearing cylindrical shell theory (also identified as sandwich cylindrical shell theory), the w^a term and the trigonometric terms are absent. As in plate theory, u^a and v^a are usually identified as β_x and β_y, respectively. In classical cylindrical shell theory, the w^a term and the trigonometric terms are absent and u^a/h and v^a/h are replaced by $-w^0_{,x}$ and $-w^0_{,y}$, respectively.

Using Eqs. (27) and (45) with θ replaced by y, the equations of equilibrium (44) become

$$N_{x,x} + N_{xy,y} = 0$$

$$N_{y,y} + N_{xy,x} + \frac{N_{yz}}{R} + \frac{N_y(w^0_{,y} - v^0/R)}{R} + \frac{N_{xy}w^0_{,x}}{R} = 0$$

$$N_{xz,x} + N_{yz,y} - \frac{N_y}{R} + \left(N_x w^0_{,x}\right)_{,x} + \left[N_y\left(w^0_{,y} - v^0/R\right)\right]_{,y} + \left(N_{xy}w^0_{,x}\right)_{,y} + \left[N_{xy}\left(w^0_{,y} - v^0/R\right)\right]_{,x} + p = 0$$

$$M_{x,x} + M_{xy,y} - N_{xz} = 0$$

$$M_{y,y} + M_{xy,x} - N_{yz} + M_{yz}/R = 0$$

$$M_{xz,x} + M_{yz,y} - N_z - M_y/R = 0$$

$$\mathscr{M}_{x,x} + \mathscr{M}_{xy,y} - \mathscr{N}_{xz} = 0$$

$$\mathscr{M}_{y,y} + \mathscr{M}_{xy,x} - \mathscr{N}_{yz} + \mathscr{M}_{yz}/R = 0$$

$$\mathscr{N}_{xz,x} + \mathscr{N}_{yz,y} + \mathscr{P}_z - \mathscr{N}_y/R = 0 \tag{48}$$

and the boundary conditions are the same as for the flat plate except for the w^0 conditions, which are, at $x = 0, a$:

$$\left[N_{xz} + N_x w^0_{,x} + N_{xy}\left(w^0_{,y} - v^0/R\right)\right] w^0 = 0 \tag{49a}$$

and at $y = 0, b$:

$$\left[N_{yz} + N_y\left(w^0_{,y} - v^0/R\right) + N_{xy}w^0_{,x}\right] w^0 = 0 \tag{49b}$$

Consider a cylinder buckling under external uniform lateral pressure and uniform axial compression. With $w^0 = \text{const}$, $v = 0$, and the deformations independent of y prior to buckling, let

$$N_x = -N^b_x \qquad N_y = -p^b R \tag{50}$$

where N^b_x and p^b are constants that depend on the magnitude of the axial load and lateral pressure, respectively. The differential equations of equilibrium can be converted into buckling equations as for flat plates by considering small additional deformations and forces at buckling. The buckling equations are

$$N_{x,x} + N_{xy,y} = 0$$

$$N_{y,y} + N_{xy,x} + N_{yz}/R - p^b\left(w^0_{,y} - v^0/R\right) = 0$$

$$N_{xz,x} + N_{yz,y} - N_y/R - N^b_x w^0_{,xx} - p^b R\left(w^0_{,y} - v^0/R\right)_{,y} = 0$$

$$M_{x,x} + M_{xy,y} - N_{xz} = 0$$

$$M_{y,y} + M_{xy,x} - N_{yz} + M_{yz}/R = 0$$

$$M_{xz,x} + M_{yz,y} - N_z - M_y/R = 0$$

$$\mathscr{M}_{x,x} + \mathscr{M}_{xy,y} - \mathscr{N}_{xz} = 0$$

$$\mathscr{M}_{y,y} + \mathscr{M}_{xy,x} - \mathscr{N}_{yz} + \mathscr{M}_{yz}/R = 0$$

$$\mathscr{N}_{xz,x} + \mathscr{N}_{yz,y} + \mathscr{P}_z - \mathscr{N}_y/R = 0 \qquad (51)$$

Note that the unknowns must now be redefined to be additional deformations and forces at buckling.

Discussion

Nonlinear equations of equilibrium and variationally consistent boundary conditions are derived for laminated and thick flat plates and cylindrical shells from three-dimensional nonlinear elasticity theory using displacements that vary trigonometrically through the thickness. From these equations, buckling equations are written for certain simple loadings. The method used is formal rather than intuitive and, by using the orthogonal curvilinear coordinates as presented here, the strains developed here can be used to derive nonlinear equations for many other kinds of shells. The equations so derived can be reduced to classical equations and to conventional transverse shearing equations.

In the present derivation, the exact strains from nonlinear elasticity are expanded into series containing only linear and second-degree terms. Thus, nonlinear theory of the von Kármán type is presented. Displacements are expanded in a series composed of a few algebraic terms and an infinite set of trigonometric terms. The algebraic terms retained allow reduction of the equations to classical equations and to conventional transverse shearing equations. The addition of the first trigonometric term permits sufficient through-the-thickness variation of displacements to allow reasonably accurate analysis of laminated and thick plates and shells with the potential energy method (virtual work). If additional accuracy is needed to determine transverse shearing and direct stress, the complementary energy method, which satisfies equilibrium through the thickness, can be used with essentially the same terms. Some assumptions of lesser importance are hidden in the analysis. Some of the conditions of rigid-body translation and rotation without strain are violated and overall changes in volume are not accounted for.

For cylindrical shells, an additional algebraic term that has been omitted in previous transverse shearing theories is included in the normal displacement. It appears in the linear part of the strains in much the same manner as the corresponding in-plane terms.

Conclusions

Nonlinear equations of equilibrium and buckling equations that include the effects of transverse shearing are derived for laminated and thick plates and shells. The derivation starts from nonlinear three-dimensional elasticity theory and, from the strains expressed in orthogonal curvilinear coordinates, nonlinear equations of equilibrium for other shells and other approximations can be derived. In addition to the usual algebraic through-the-thickness terms assumed for the displacements, trigonometric through-the-thickness terms are added to give more accurate results. The equations can be reduced to conventional transverse shearing theory and to classical theory when the trigonometric terms are omitted. Numerical results presented here for aluminum plates show that for thicker plates, classical theory gives buckling stresses that are unconservative.

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